Theoretical considerations about utility and uncertainty

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Abstract

Before making a decision for an uncertain problem, we should build a functional model with a view of evaluating the level of satisfaction of the decision maker who undertakes the risk. If such a functional model is obtained, the decisional problems can be solved by looking for a decision able to maximize the decision maker's level of satisfaction. The above concepts can be used in the insurance field because any individual or a business entity can choose, at a certain moment, between a version of insurance x appreciated with a probability α and another version of insurance y with a probability $(1-\alpha)$ and with utilities U(x) and respectively U(y). The individual or business entity in question will prefer insurance x over insurance y if he/she/it is convinced that, from the point of view of the utility of the action, insurance x is more useful, which means that x>y when U(x)>U(y) and vice versa. Several empirical tests proved that a portfolio is sufficiently diversified if it includes 20-30 titles, while beyond this number, the marginal decrease of the specific risk is insignificant, and lower, anyway, than the costs incurred. The total risk of an insurance portfolio cannot be decreased by diversification beyond the limit of 30-40%, which represents the percentage of the market risk of the portfolio as compared to the total risk of the titles.

Keywords: utility, uncertainty, risks, portfolio, insurance, marginal decrease

JEL Classification: C21, G22, G23

Utility theory

Webster's Dictionary (1990 edition) defines risk as "possibility of loss or failure, hazard, danger, jeopardy". Le Petit Larousse Illustré (1995 edition) mentions two possible acceptable meanings for the notion of risk: "hazard, more or less probable disadvantage to which one is exposed" on one hand, and on the other hand, "prejudice, possible disaster covered by an insurance company". The Small Romanian Language Dictionary (1978 edition) defines risk in the same manner, as "hazard, possible disadvantage". In relation to these definitions, we would like to make two comments:

- in all the definitions, the notion of risk has negative connotations, since it refers to the chance or likelihood of the future occurrence of an unwanted event, with a negative impact on a person, on an organization or on the whole society. However, we would like to add that taking risks is not necessarily a bad thing, and that avoiding risks is not necessarily wise. Peter Drucker, a reputed management specialist, wrote in a paper that: “People who don't take risks generally make about two big mistakes a year. People who do take risks generally make about two big mistakes a year.” Thus, the interaction can involve the same risks as the action (sometimes higher).

- The notion of risk is closely related to the *likelihood* of the occurrence events with unfavourable consequences. Where there is risk, the consequences of an action cannot be predicted accurately. The exposure to risk is created whenever an action may give rise to a loss or a profit that cannot be foreseen.

The fundamental objective of the theory of utility under conditions of uncertainty is the rationalisation of the choices made by business entities in risky situations of the insurance market, among others. Economics studies the way in which people choose among versions of allocation of limited resources fro the wealth distribution in time. Economic theory admits that there are differences among consumers’ tastes, but has few things to say about the reason of their existence or cause.

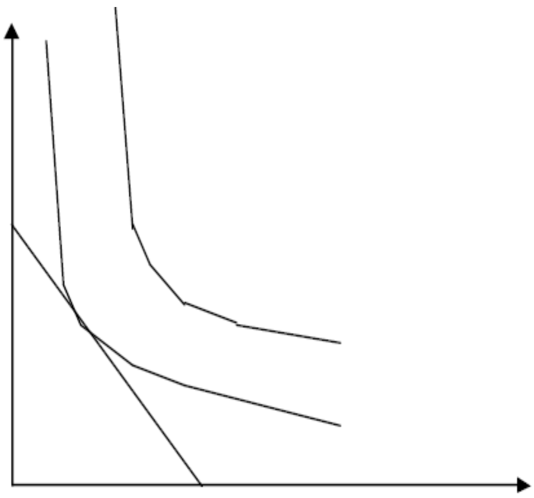

Figure nr. 1a)
Choosing among convenience goods under conditions of certainty

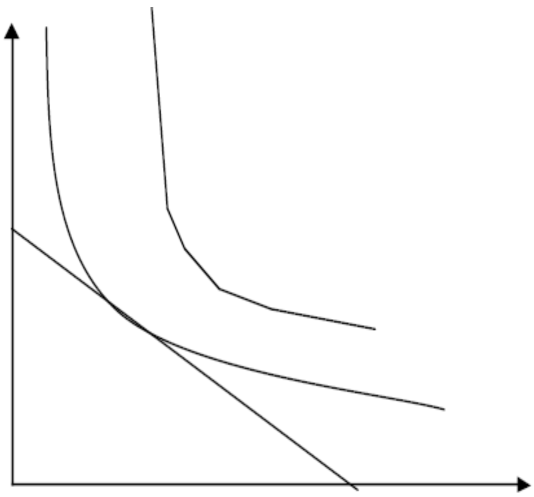

Figure nr.1b)
Choosing between consumption and investment under conditions of certainty

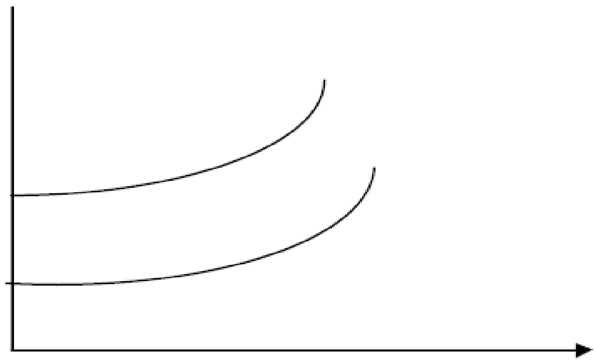

Figure nr. 1c) Indifference curves concerning the decision-making under uncertainty

There are also other behavioural theories providing details concerning the choice theory such as: social sciences, psychology, political sciences, sociology, etc. Nevertheless, there is much to say about the theory of choice under uncertainty, for example why the risk aversion is lower in a 60-year old person, as compared to the same person at the age of 25, or why some people prefer meat products while others prefer vegetables.

The theory of the investor’s choice is part of what came to be known as the utility theory.

The price theory of microeconomics studies the choices made concerning interchangeable goods such as apples and oranges at the same time. The resulting indifference curves are shown in fig. 1a. Another type of choice available to individuals is the choice between spending now or preserving (investing) and spending more in the future. This is the theory of choosing the time. This type of decision concerning the spending period (investments) is shown in fig. 1b. Our main concern is to choose among risk alternatives which do not depend on the period of time, and is emphasized in the theory of the investor’s choice. Graph 1c represents the indifference curves concerning the decision-making under uncertainty, without periods of time.

The first problem that raised scientists’ interest in relation to investment decisions under uncertainty was the so-called “St Petersburg paradox“.

This is enounced as follows: “While gambling, an individual tosses a fair coin until the first “head” is scored. If this “head” is scored at the n^{th} trial, than the individual will give money to another person”.

If this “head” is scored at the n^{th} trial, the gambler will pay a sum $S_n=2^{n-1}$u.m to another person. Given the fact that the probability to score a “head” at one trial is ½, the probability to obtain the “head” at the n^{th} trial is $p_0=(1/2)^n$. Under these circumstances, the mathematical expectation to win the bet C in this gamble is as follows:

$$E(C)=\sum_{n=1}^{\infty} p_n S_n=\sum_{n=1}^{\infty}\frac{1}{2^n}*2^{n-1}=\sum^{\infty}\frac{1}{2}=\infty$$

The paradox consists of the fact that no realistic person will view this winning as infinite, and will limit it to maximum 10-15 m.u. In order to solve this paradox, we use the concept of its utility. For example, according to this theory, any individual taking part into this game would not appreciate the value won in itself, but the expected utility of this sum. Using the logarithmic function as utility function, Bernoulli proved that the expected utility value of the winning is a finite value (Bernoulli, 1954, p.23).

The description of an uncertain environment contains two different types of information. First of all, all the possible outcomes should be mentioned. An outcome is a list of variables which influence the decision maker’s mood. The list may make reference to someone’s health condition, certain methodological parameters, the levels of pollution, or quantities of various consumed goods.

As long as we do not save, we will assume that the outcome can be measured using a uni-dimensional unit, namely money (used at a certain moment). For example, we will assume that X is a set of possible outcomes. In order to avoid the technical details, we will also assume that the number of outcomes is finite, consequently $X=\{x_s\}s=1,\ldots,n$

The second aspect that characterises an uncertain environment is the vector of the probabilities for each possible outcome. We will assume that $p_s>0$ is the probability of occurrence of x_s, with $\sum_{s=1}^{n} p_s = 1$. A lottery L for a vector (x_1, p_1; x_2, p_2; x_n, p_n) or for a random variable $\begin{pmatrix} x_1 x_2 \ldots\ldots x_n \\ p_1 p_2 \ldots\ldots p_n \end{pmatrix}$. Given the fact that the set of outcomes is invariant, we will define a lottery by its probability vector ($p_1 \ldots\ldots\ldots p_n$).

The set of all the lotteries made in relation to the outcomes x is given by the following:

$$E(C) = \sum_{n=1}^{\infty} p_n S_n = \sum_{n=1}^{\infty} \frac{1}{2^n} * 2^{n-1} = \sum^{\infty} \frac{1}{2} = \infty$$

When n=3, we can represent a lottery by a point $(p_1 p_3) \in R^2$ in which $p_2 = 1 - p_1 - p_3$. More precisely, in order to be a lottery, this point should be included in the so-called Machina triangle $\{(p_1 p_3) \in R_+^2 / 1 - p_1 - p_3 \leq 1\}$ (Machina, 1987, p.3). If the lottery is on one edge of the triangle, one of the probabilities is zero. If the lottery is at a corner, this lottery is degenerated, which means that it takes one of the values x_1, x_2, x_3 with probability 1. This triangle is shown in fig. 2.

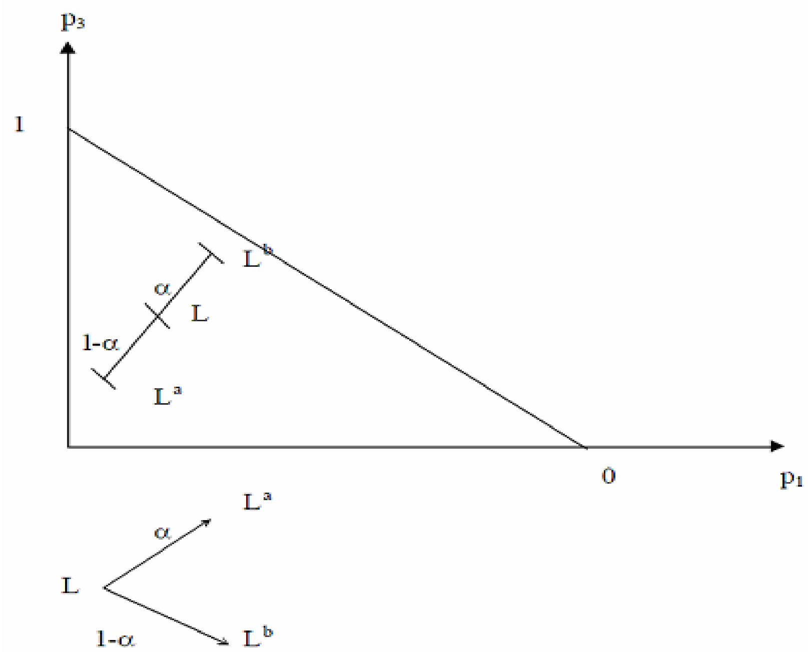

Figure nr. 2 The Machina triangle

A compound lottery is a lottery whose outcomes are, in their turn, lotteries. Let us consider a compound lottery L which contains lottery

$L^a = \left(p_1^a \ldots\ldots p_n^a \right)$ with probability α, and lottery $L^b = \left(p_1^b \ldots\ldots p_n^b \right)$, with probability $1-\alpha$. The probability that the outcome of L be x1 is $p_1 = \alpha\, p_1^a + (1-\alpha)\, p_1^b$. More generally, we find out that L has the same vector of probabilities as $\alpha L^a + (1-\alpha) L^b$.

A compound lottery is a convex combination of simple lotteries.

Irrespective of the nature of its preoccupations, the evaluation of the consequences or outcomes of the various actions performed by an individual or a business entity implies two extremely important problems, namely:

1 how to measure the outcome of the actions performed
2 how to assess these outcomes from the perspective of the measurements performed

By associating the outcomes of his/her actions with certain numbers, an individual will not always be able to assess the value of these outcomes as well, just by mere measurements. He/she should also associate these outcomes with other numbers as well, irrespective of the "size" of these outcomes, by which he/she should be able to make yet another assessment of the outcomes, in addition of the dimensional one. Such relevant numbers or the utility of such outcomes are simply referred to as utilities.

We define S as the set of the uncertain alternatives, and on this set we induce a binary operation "ϕ" determined by the axioms on which the individuals' rational and consistent behaviour is based. These represent the main axioms on which the cardinal utility theory is based. Even if a whole structure was built on them, culminating with the theory of the efficient capital markets, and with the models used for the evaluation of the assets on these markets, they contain a certain degree of contradiction with the real world, an aspect which is emphasized by the paradox discovered by the economist Maurice Allais (Conlisk, 1989, p. 392). For example, if we have two lotteries: the first of these two lotteries consisting of two gambles "a" and "b", and the second of two gambles "i" and "j", as in the figure below:

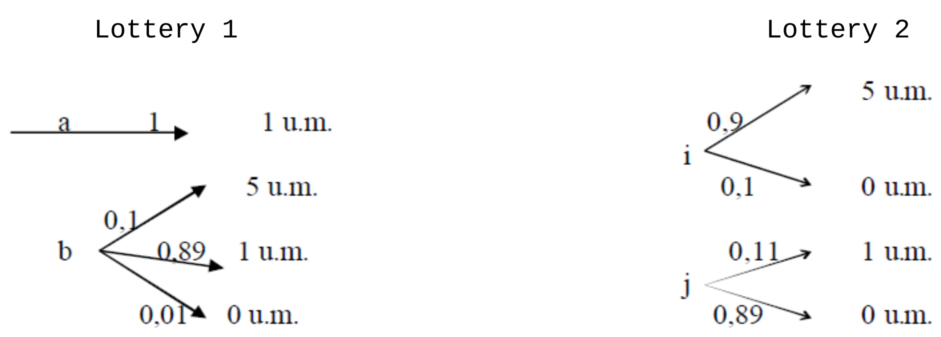

Figure nr. 3 The Allais paradox

Obviously, one investor with risk aversion will choose version "a" of lottery 1, and version "i" of lottery 2.

$$a \approx 1.1u.m. \approx 0{,}11.1u.m. + 0{,}89.1u.m. \approx 0{,}11(1.1u.m.) + 0{,}89.1u.m.$$

$$b \approx 0{,}1.5u.m. + 0{,}89.1u.m. + 0{,}01.0u.m. \approx 0{,}11\left(\frac{1}{11}.0u.m. + \frac{10}{11}.5u.m.\right) + 0{,}89.1u.m.$$

$$a \succ b \Rightarrow 0{,}11.(1.1u.m.) + 0{,}89.1u.m. \succ 0{,}11\left(\frac{1}{11}.0u.m. + \frac{10}{11}.5u.m.\right) + 0{,}89.1u.m.$$

From the formula above, we obtain that $1.1u.m. \succ \frac{1}{11}.0u.m. + \frac{10}{11}.5u.m.$

Consequently, from the substitution axiom, the formula would no longer be valid, due to the fact that the axiom is verified for each property. By applying the substitution axiom again to the last formula, we will obtain:

$$0{,}11.(1.1u.m.) + 0{,}89.0u.m. \succ 0{,}11\left(\frac{1}{11}.0u.m. + \frac{10}{11}.5u.m.\right) + 0{,}89.0u.m.$$

From which $0{,}11.1u.m. + 0{,}89.0u.m. \succ 0{,}1.5u.m. + 0{,}9.0u.m.$

Which means that $j \succ i$, which is in contradiction with reality.

The four axioms represent the basis of the capital market financial theory, because, based on the preference relation described by them we can build a cardinal utility function U. when, for two alternatives x and y, $x \succ y$, then U(x)>U(y), which means that utility is monotonically increasing as compared to the preference, and $U(\alpha x + (1-\alpha)y) = \alpha U(x) + (1-\alpha)U(y)$.

This signifies that the utility of an alternative is equal to the average value of the utilities of the potential outcomes of this alternative, and then function U(x) is called a utility function.

When U is a utility function and V=at+b, a>0, a linear function, then the composite function $V \circ U$ is also a utility function, because:

V(U(x))=aU(x)+b

Fie $x \succ y \Rightarrow U(x) > U(y) \Rightarrow aU(x) + b > aU(y) + b \Rightarrow V(U(x)) > V(U(y))$

$$V(U(x,\alpha;y,1-\alpha)) = aU((x,\alpha;y,1-\alpha)) + b = a[\alpha U(x) + (1-\alpha)U(y)] + b =$$

$$\alpha[aU(x) + b] + (1-\alpha)[aU(y) + b] = \alpha V(U(x)) + (1-\alpha)V(U(y))$$

Model application in insurance

The above concepts can be used in the insurance field because any individual or a business entity can choose, at a certain moment, between a version of insurance x appreciated with a probability α and another version of insurance y with a probability $(1-\alpha)$ and with utilities U(x) and respectively U(y). The individual or business entity in question will prefer insurance x over insurance y if he/she/it is convinced that, from the point of view of the utility of the action, insurance x is more useful, which means that x>y when U(x)>U(y) and vice versa.

If a spending plan X is made of n risky alternatives x_i with probabilities p_i, then $E(U(X)) = \sum_{i=1}^{n} p_i x_i$. From all the experiments or lotteries that might occur with probabilities p_i, and utilities U_i, the best lottery is the one for which the average associated utility is maximum.

The expected value may represent a selection criterion only for large portfolios of identical and independent risks. The insurance selection may be made by the maximization of the expected value. The expected value criterion may be successfully used in the case of casco insurances, due to the fact that, in the case of this type of insurance, the risks are equal and independent.

In the case of an insurance company who concludes such insurance policies, the average probability of risk occurrence is 35%, while the average value of the damage is 8,000,000 m.u. The lottery associated to this type of insurance is the following:

$$p_s = 0{,}35 \qquad \overline{X} = -8000000$$
$$P_s = 0{,}65 \qquad \overline{X} = 0$$

The value loss caused by a disaster will be equal to the expected value of the lottery, while the average risk of the occurrence of this damage is as follows:

$$\sigma^2(X) = 0{,}35\left[-2{,}8-(-8)\right]^2 + 0{,}65\left[0-(-2{,}8)\right]^2$$
$$\sigma^2(X) = 9{,}46 + 5{,}09 = 14{,}55\,mil$$
$$\sigma(X) = 3{,}8mil$$

Consequently, the annual average loss following the occurrence of a risk is 3.8 million. Thus an insurance holder should pay an annual value of 3.8 million to the company, if it has only one insurance holder. In a few years, the insurance holder comes to pay, as an insurance premium, the whole value of his car.

This sacrifice might prove to be useless because, in this period, the risk might not occur. Let us assume that the insurance company has managed to conclude an insurance contract of this type, and the risk occurrence likelihood for an insurance holder does not depend on his/her registration to another. At the same time, we will assume that the loss is the same as the equi-weighted portfolio damage, namely:

$$E(p) = \sum \frac{1}{n} E(x_i) = E(x_i)$$
$$E(p) = 3{,}8mil$$
$$\sigma^2(p) = \frac{1}{n^2}\sigma^2(x_i) = \frac{\sigma^2(x_i)}{n}$$
$$\sigma^2(p) = \frac{1455}{n}$$

The higher the number of concluded insurance policies, the higher the risk of the insurance policy.

Consequently, the risk is greatly decreased, and the insurance company has the possibility to decrease the insurance premium, this making the insurance policy more accessible to the insurance holders.

Consequently, by diversification, the risk of the insurance portfolio is significantly decreased.

In our case, for a risk of $\sigma^2(x_0)=14.55$ for a monthly insurance, the total risk of the portfolio thus is developed function of the number of concluded policy insurances.

Table 1: Risk evolution function of the number of insurance policies

n	$\sigma^2 p$	σp
1	14,55	3,8
2	14,55/2=7,2	2,69
-	-	-
10	14,55/10=1,45	1,2
-	-	-
20	14,55/20=0,7275	0,85
. . . 50	. . . 14,55/50=0,291	. . . 0,53
. .	. .	. .
100	14,55/100=0,14	0,37

Several empirical tests proved that a portfolio is sufficiently diversified if it includes 20-30 titles, while beyond this number, the marginal decrease of the specific risk is insignificant, and lower, anyway, than the costs incurred.

The total risk of an insurance portfolio cannot be decreased by diversification beyond the limit of 30-40%, which represents the percentage of the market risk of the portfolio as compared to the total risk of the titles.

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